Incompatibility of quantum instruments

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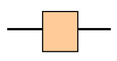
Results – arXiv:2212.11225

Abstract

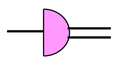
Quantum instruments describe outcome probability as well as state change induced by measurement of a quantum system. Incompatibility of two instruments, i. e. the impossibility to realize them simultaneously on a given quantum system, generalizes incompatibility of channels and incompatibility of positive operator-valued measures (POVMs). We derive implications of instrument compatibility for the induced POVMs and channels. We also study relation of instrument compatibility to the concept of non-disturbance. Finally, we prove equivalence between instrument compatibility and postprocessing of certain instruments, which we term complementary instruments. We illustrate our findings on examples of various classes of instruments.

Notation

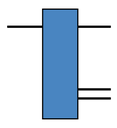
Notation of the set:



Channel = CPTP map $\mathrm{Ch}(\mathcal{H},\mathcal{K})$



POVM = Positive operator valued measure $\mathcal{O}(\Omega,\mathcal{H})$



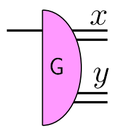
Instrument = collection of CPTD maps summing up to a channel $\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$

$\Phi^{\mathcal{G}}$, $\mathsf{A}^{\mathcal{G}}$ **induced channel, POVM** of instrument $\mathcal{G}$, respectively

Compatibility of POVMs and channels

Compatibility of POVMs $\mathsf{A}\in\mathcal{O}(\Omega,\mathcal{H})$ $\mathsf{B}\in\mathcal{O}(\Lambda,\mathcal{H})$

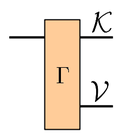
$\mathsf{A}\circledcirc\mathsf{B}\Leftrightarrow\exists\,\mathsf{G}\in\mathcal{O}(\Omega\times\Lambda,\mathcal{H})$



$$\sum_{y\in\Lambda}\mathsf{G}(x,y)=\mathsf{A}(x)\quad\forall x\in\Omega$$
$$\sum_{x\in\Omega}\mathsf{G}(x,y)=\mathsf{B}(y)\quad\forall y\in\Lambda$$

Compatibility of channels $\Phi\in\mathrm{Ch}(\mathcal{H},\mathcal{K})$ $\Psi\in\mathrm{Ch}(\mathcal{H},\mathcal{V})$

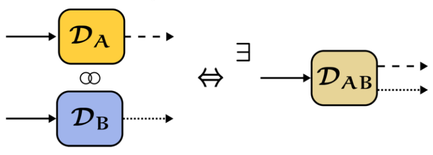
$\Phi\circledcirc\Psi\Leftrightarrow\exists\,\Gamma\in\mathrm{Ch}(\mathcal{H},\mathcal{K}\otimes\mathcal{V})$



$$\mathrm{tr}_{\mathcal{V}}\left[\Gamma(\varrho)\right]=\Phi(\varrho)$$
$$\mathrm{tr}_{\mathcal{K}}\left[\Gamma(\varrho)\right]=\Psi(\varrho)$$

Definition of Instrument compatibility

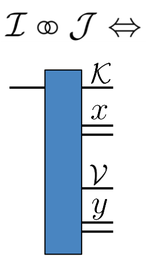
The starting point for our work is what can be called operational description of incompatibility:



compatible devices should allow for joint simultaneous implementation, meaning that a joint device with a single input should be used to obtain the correct output for each original device at each experimental run. For instruments this is captured by the notion of parallel compatibility [1], and it generalizes notions of compatibility of quantum channels and POVMs.

Compatibility of instruments $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$

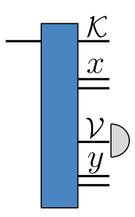
$\mathcal{I}\circledcirc\mathcal{J}\Leftrightarrow\exists\,\mathcal{G}\in\mathrm{Ins}(\Omega\times\Lambda,\mathcal{H},\mathcal{K}\otimes\mathcal{V})$



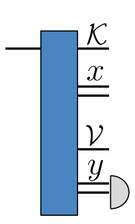
$$\sum_{y\in\Lambda}\mathrm{tr}_{\mathcal{V}}\left[\mathcal{G}_{(x,y)}(\varrho)\right]=\mathcal{I}_x(\varrho)\quad\forall x\in\Omega$$
$$\sum_{x\in\Omega}\mathrm{tr}_{\mathcal{K}}\left[\mathcal{G}_{(x,y)}(\varrho)\right]=\mathcal{J}_y(\varrho)\quad\forall y\in\Lambda$$

Direct definition consequences

We find that compatibility with an instrument automatically implies compatibility with its induced POVM and its induced channel. These necessary conditions might be often easier to check, and would quite often suffice to conclude incompatibility of a given pair of instruments.

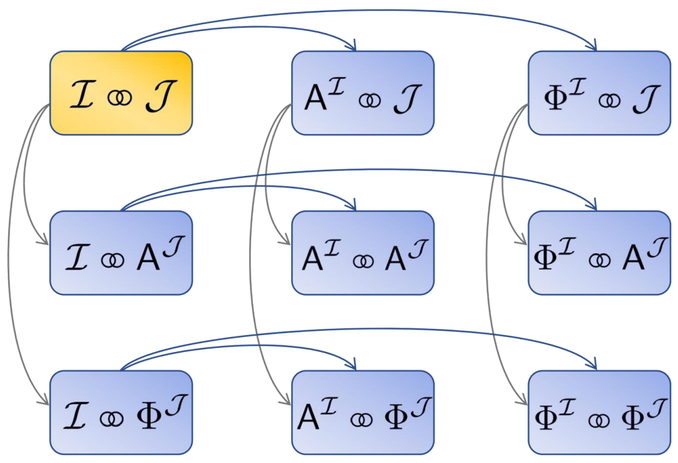


$$\mathcal{I}\circledcirc\mathcal{J}\Rightarrow\mathcal{I}\circledcirc\mathsf{A}^{\mathcal{J}}$$



$$\mathcal{I}\circledcirc\mathcal{J}\Rightarrow\mathcal{I}\circledcirc\Phi^{\mathcal{J}}$$

Implications of the compatibility of instruments $\mathcal{I}$ and $\mathcal{J}$

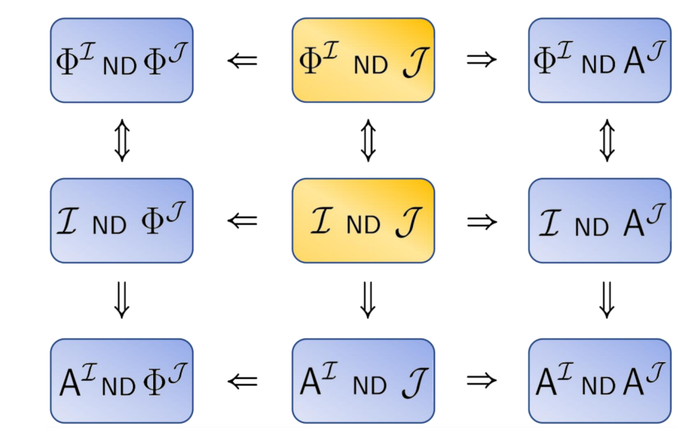


Definition: Non-disturbance for q. Instruments

An instrument $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ is said not to disturb instrument $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$ if $\mathcal{J}_y\circ\Phi^{\mathcal{I}}=\mathcal{J}_y$ for all $\forall y\in\Lambda$. We also say that a POVM $\mathsf{A}\in\mathcal{O}(\Omega,\mathcal{H})$ does not disturb an instrument $\mathcal{J}$ if there exists an instrument $\mathcal{I}$ with $\mathsf{A}^{\mathcal{I}}=\mathsf{A}$ such that $\mathcal{I}$ does not disturb $\mathcal{J}$.

$$\mathcal{I}\ \mathrm{ND}\ \mathcal{J}\Leftrightarrow\mathcal{J}\circ\Phi^{\mathcal{I}}=\mathcal{J}$$

Implications of $\mathcal{I}$ ND $\mathcal{J}$ for induced POVM/channel



Non-disturbance of an instrument automatically implies non-disturbance of its induced POVM and its induced channel.

Proposition 1. If a POVM A does not disturb an instrument $\mathcal{J}$, then A and $\mathcal{J}$ are compatible. Furthermore, for sharp A also the converse holds.

Corollary

$\mathcal{I}\ \mathrm{ND}\ \mathcal{J}\ \Rightarrow\ \mathsf{A}^{\mathcal{I}}\ \mathrm{ND}\ \mathcal{J}\ \Rightarrow\ \mathsf{A}^{\mathcal{I}}\circledcirc\mathcal{J}\ \Rightarrow\mathsf{A}^{\mathcal{I}}\circledcirc\Phi^{\mathcal{J}}$ and $\mathsf{A}^{\mathcal{I}}\circledcirc\mathsf{A}^{\mathcal{J}}$

Post-processing of quantum instruments

An instrument $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ is a postprocessing of instrument $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$, denoted by $\mathcal{I}\preceq\mathcal{J}$, if there exist instruments $\{\mathcal{R}^{(y)}\}_{y\in\Lambda}\subset\mathrm{Ins}(\Omega,\mathcal{V},\mathcal{K})$ such that $\mathcal{I}_x=\sum_{y\in\Lambda}\mathcal{R}^{(y)}_x\circ\mathcal{J}_y$ for all $x\in\Omega$.

Proposition 2. Let $\mathcal{I}$ and $\mathcal{J}$ be two instruments. If $\mathcal{I}\preceq\mathcal{J}$, then $\mathcal{I}\circledcirc\mathsf{A}^{\mathcal{J}}$, and in particular $\Phi^{\mathcal{I}}\circledcirc\mathsf{A}^{\mathcal{J}}$ and $\mathsf{A}^{\mathcal{I}}\circledcirc\mathsf{A}^{\mathcal{J}}$.

Complementary quantum instrument

Complementary quantum channel. Each Stinespring dilation $(\mathcal{H}_A,W)$ of a channel $\Phi\in\mathrm{Ch}(\mathcal{H},\mathcal{K})$ defines a complementary channel (or conjugate channel) $\Phi^C\in\mathrm{Ch}(\mathcal{H},\mathcal{H}_A)$ by

$$\Phi^C(\varrho)=\mathrm{tr}_{\mathcal{K}}\left[W\varrho W^*\right]$$

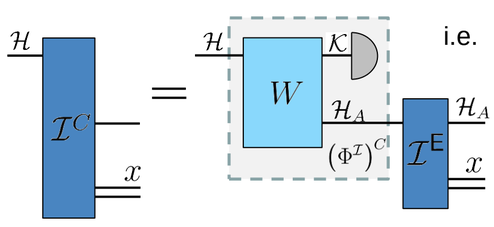
Lüders instrument. For a POVM $\mathsf{A}\in\mathcal{O}(\Omega,\mathcal{H})$ we define Lüders instrument $\mathcal{I}^{\mathsf{A}}\in\mathrm{Ins}(\Omega,\mathcal{H})$ of A by setting $\mathcal{I}^{\mathsf{A}}_x(\varrho)=\sqrt{\mathsf{A}(x)}\varrho\sqrt{\mathsf{A}(x)}$.

Dilation of an instrument $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ is a triple $(\mathcal{H}_A,W,\mathsf{E})$ consisting of a Hilbert space $\mathcal{H}_A$, an isometry $W:\mathcal{H}\to\mathcal{H}_A\otimes\mathcal{K}$ and a POVM $\mathsf{E}\in\mathcal{O}(\Omega,\mathcal{H}_A)$ such that

$$\mathcal{I}_x(\varrho)=\mathrm{tr}_{\mathcal{H}_A}\left[W\varrho W^*\left(\mathsf{E}(x)\otimes I_{\mathcal{K}}\right)\right]$$

Complementary instrument. For instrument $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ with a dilation $(\mathcal{H}_A,W,\mathsf{E})$ we define complementary instrument $\mathcal{I}^C\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{H}_A)$ relative to this dilation via the formula

$$\mathcal{I}^C_x(\varrho)=\mathrm{tr}_{\mathcal{K}}\left[\left(\sqrt{\mathsf{E}(x)}\otimes I_{\mathcal{K}}\right)W\varrho W^*\left(\sqrt{\mathsf{E}(x)}\otimes I_{\mathcal{K}}\right)\right]$$



i.e. $\mathcal{I}^C_x=\mathcal{I}^{\mathsf{E}}_x\circ\left(\Phi^{\mathcal{I}}\right)^C$

Bibliography:

[1] A. Mitra and M. Farkas, Compatibility of quantum instruments, **Phys. Rev. A 105, 052202 (2022).**

[2] A. Mitra and M. Farkas, Characterizing and quantifying the incompatibility of quantum instruments, **arXiv:2209.02621 (2022).**

[3] L. Leppäjärvi and M. Sedlák, Postprocessing of quantum instruments, **Phys. Rev. A 103, 022615 (2021).**

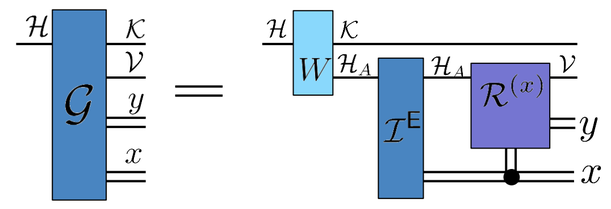
Proposition 3 All complementary instruments of instrument $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ are postprocessing equivalent, i.e. let $(\mathcal{H}_A,W,\mathsf{E}),(\mathcal{H}'_A,W',\mathsf{E}')$ be two dilations of $\mathcal{I}$, then $\mathcal{I}^C\preceq\mathcal{I}^{C'}$ and $\mathcal{I}^{C'}\preceq\mathcal{I}^C$.

Main result: Characterization of instrument incompatibility in terms of postprocessing

Theorem 1: Let $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ and $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$ be instruments and let $\mathcal{I}^C\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{H}_A)$ be complementary instrument related to any dilation $(\mathcal{H}_A,W,\mathsf{E})$. Then the following are equivalent:

i) $\mathcal{I}\circledcirc\mathcal{J}$,
ii) $\mathcal{J}\preceq\mathcal{I}^C$.

Thus, in general:



Corollaries of Theorem 1:

Corollary 1: Let $\mathsf{A}\in\mathcal{O}(\Omega,\mathcal{H})$ be POVM, $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$ an instrument. Then $\mathsf{A}\circledcirc\mathcal{J}$ if and only if $\mathcal{J}\preceq\mathcal{I}^{\mathsf{A}}$.

Corollary 2: Let $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$, $\mathsf{B}\in\mathcal{O}(\Lambda,\mathcal{H})$ and $\mathcal{J}^{\mathsf{B}}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$ some measure-and-prepare instrument which measures B. Then $\mathcal{I}\circledcirc\mathcal{J}^{\mathsf{B}}$ if and only if $\mathcal{I}\preceq\mathcal{I}^{\mathsf{B}}$.

Proposition 4: Two measure-and-prepare instruments are compatible if and only if their induced POVMs are compatible.

Lemma 1. For indecomposable instrument $\mathcal{I}$ there exists a dilation such that the complementary instrument $\mathcal{I}^C$ is a measure&prepare instrument $\mathcal{I}^C_x(\varrho)=\mathrm{tr}\left[\mathsf{A}^{\mathcal{I}}(x)\varrho\right]|\varphi_x\rangle\langle\varphi_x|$ for some orthonormal basis $\{\varphi_x\}_{x\in\Omega}$ of $\mathcal{H}_A$.

Corollary 3. Let $\mathcal{I}\in\mathrm{Ins}(\Omega,\mathcal{H},\mathcal{K})$ be an indecomposable instrument. Then an instrument $\mathcal{J}\in\mathrm{Ins}(\Lambda,\mathcal{H},\mathcal{V})$ that is compatible with $\mathcal{I}$ must be of the form

$$\mathcal{J}_y(\varrho)=\sum_{x\in\Omega}\mathrm{tr}\left[\nu_{xy}\mathsf{A}^{\mathcal{I}}(x)\varrho\right]\xi_{xy}$$

for some family of states $\{\xi_{xy}\}_{x\in\Omega,y\in\Lambda}\subset\mathcal{S}(\mathcal{V})$ and some stochastic post-processing matrix $\nu:=(\nu_{xy})_{x\in\Omega,y\in\Lambda}$.

Proposition 5: Two indecomposable instruments are compatible if and only if their induced POVMs are postprocessing equivalent rank-1 POVMs. Furthermore, in this case both of the instruments are measure-and-prepare instruments which prepare pure states.

Summary

The aim of this work was to investigate incompatibility of instruments and to find links of this problem to other problems and known results. As special cases we consider compatibility of a POVM with another POVM / channel / instrument, or compatibility of a channel with another channel / instrument as a special case of compatibility of two instruments. We consider two instruments to be compatible, if there exists a joint instrument, whose single input and two outputs are the same as for the two instruments and each of these two original instruments can be obtained via classical and quantum marginalization over one of the outputs. In other words, we advocate for definition of compatibility of instruments that first appeared in the work of Mitra and Farkas [1] under the name parallel compatibility to become a standard. In this way, the newly obtained results either coincide with the known results in the special cases or they can bring entirely new results also for them.

Compatibility of POVMs is known to be closely related to the concept of non-disturbance. We generalize this concept from POVMs and define that an instrument does not disturb another instrument if the concatenation of the induced channel of the first instrument with the second instrument is the same as (the direct use of) the second instrument. We show that if the first instrument does not disturb the second instrument several non-disturbing relations hold for the induced POVMs and channels.

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